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Solitons in Transitional Boundary Layers

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Two-dimensional evolution equations are derived as applied to flows in the near-wall jet and the Blasius boundary layer on a flat plate on which a mechanism of inviscid–inviscid interaction controls the development of large-sized short-scaled disturbances. The first one is an extension of the Korteweg–de Vries equation. As distinct from the shallow-water wave motion underlying the Kadomtsev–Petviashvili equation, the fluid parameters are not assumed to depend only weakly on the direction transversal to the oncoming stream. The second dynamical system provides a two-dimensional analog of the Benjamin–Davis–Acrivos equation. Simple line-soliton solutions are presented in both cases. A generalized Hirota function allows a pair of crossed solitons to be obtained at some distance from a solid surface in the near-wall jet. An oblique periodic nonlinear wave train pointed out for the Blasius boundary layer comes in place of the Tollmien–Schlichting waves when their amplitude attains sufficiently large values.

Nomenclature

A	=	instantaneous displacement thickness
B	=	constant
c	=	phase velocity
F	=	auxiliary function
f	=	Hirota function
k	=	streamwise wave number
L	=	interaction-law operator
m	=	spanwise wave number
p	=	self-induced pressure
Q	=	auxiliary function
q	=	soliton-amplitude parameter
Re	=	Reynolds number
t	=	time
u, v, w	=	velocity components
W	=	crossflow
x, y, z	=	Cartesian coordinates
$\Delta, \delta, \varepsilon$	=	small parameters
η	=	water–wave amplitude
τ	=	skin friction
ω	=	frequency
$\bar{\ }$	=	nondimensional variables

Subscripts

w	=	wall condition
1, 2, 3	=	term order in asymptotic expansions

I. Introduction

IN INCOMPRESSIBLE, steady, initially two-dimensional boundary layers on flat plates, airfoils, and near-wall jets, transition to turbulence starts with the enhancement of linear Tollmien–Schlichting (TS) eigenmodes if the Reynolds number exceeds a certain critical value. The first signs of nonlinearity as a rule entwined with three-dimensionalization of the velocity field are fairly weak (see Schlichting [1]). The vortex loops resulting at a later stage of the breakdown of the two-dimensional TS wave trains can appear in different forms, depending on the type of flow and the strength of a perturbing source.

There exist two main paths to transition. The first one takes place when the perturbing agency is of small size. Then the disturbance pattern arises in a staggered array. This is characteristic of the *N* route to transition provoked by the parametric resonant amplification of the background subharmonic modes. The possibility for this scenario to become operative was theoretically predicted by Craik [2] and also indicated by Smith and Stewart [3] by extending an asymptotic version of the weakly nonlinear theory. Its basic concepts have been advanced by Landau and Lifshitz [4] and worked out by Stuart [5]. According to Goldstein [6,7], the growing wave amplitude obeys integral-differential equations with quadratic-to-quartic-type nonlinearities. Experimentally, the *N* route to transition was carefully studied by Kachanov et al. [8], Saric et al. [9], and Corke [10] in wind-tunnel tests with artificially excited mild perturbations.

When the strength of a perturbing source increases beyond a threshold magnitude, the *N* route to transition gives way to the *K* regime originally recorded by Schubauer and Klebanoff [11], Klebanoff and Tidstrom [12], and Klebanoff et al. [13]. The thatching arrangement of vortex loops before breakdown proves to be fully suppressed, and they emerge in well-organized streamwise rows featuring the second scenario of transition. Later measurements by Kachanov et al. [14] revealed the existence of an essentially two-dimensional stage of the TS wave amplification that occurs somewhat earlier than the disturbance-field three-dimensionalization. When the TS wave train enters this stage, the nearly harmonic shape of a signal becomes heavily distorted, and each oscillation cycle on the velocity oscilloscope traces assumes the distinctive form of a narrow large-sized spike (or flash). As was found by Kachanov [15,16], downstream of the site at which the spikes were given birth, self-focusing leads to strong variations of their parameters in all three directions.

Earlier attempts by Tani and Komoda [17] and Landahl [18] to theoretically explain the mechanism of spike generation in the *K* route to transition were based on the concept of local secondary high-frequency (LSHF) instability of shear flows. The essence of this mechanism is rooted in the rapid growth of high-frequency fluctuations induced by instantaneous inflectional mean-velocity profiles that arise and fade away locally in the flowfield under the action of a strongly amplified primary TS wave. However, Borodulin and Kachanov [19,20] showed that LSHF instability, although present, was not the immediate cause of the spike generation. According to their experimental findings, enhancing high-frequency oscillations occur in the near-wall sublayer, whereas the spikes develop in an intermediate region elevated above a solid surface. The two disturbances turn out to be inherently different and separated in space. Careful processing of observational data led Borodulin and Kachanov [19] to conclude that "Despite the existence of the rather

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strong dispersion of the instability waves. . .the spikes do not disperse but, on the contrary, focus in narrow flashes and propagate steadily downstream within the boundary layer almost without change of their shape and amplitude." A remarkable final statement made in the same paper, "It is highly probable that the behavior of the spikes. . .can be described in the context of the theory of solitons," seems to naturally follow from the preceding experimental evidence.

Recent investigations by Bake et al. [21] suggested that the formation of spikes is intrinsic to late stages in both the N and K paths to boundary-layer transition. However, the type of spike solitons has not been identified in earlier wind-tunnel tests. The first asymptotic analysis of large-sized short-scaled signals has been independently reported by Zhuk and Ryzhov [22] and Smith and Burggraf [23]. As was elucidated in extended studies by Rothmayer and Smith [24] and by Ryzhov [25], solitons belong to different dynamical systems, depending on the character of shear flow. The classical Korteweg–de Vries equation controls the instantaneous displacement thickness in the near-wall-jet solitons (see Korteweg and de Vries [26]) typical of shallow-water channels. In the Blasius boundary layer, the evolution of solitons obeys an integral-differential equation introduced into mathematical physics by Benjamin [27] and Davis and Acrivos [28] (BDA) in connection with deepwater internal waves. In line with the experimental findings by Borodulin and Kachanov [19,20], the large-amplitude disturbances originate in a specific adjustment zone sandwiched between the main body of a boundary layer and the viscous near-wall sublayer, with the inviscid–inviscid interaction being a mechanism driving these nonlinear signals. Quantitative comparisons of the theoretical and experimental results related to the K route to transition in the Blasius boundary layer were exposed by Kachanov et al. [29]. The most decisive conclusion comes from the spectral decomposition of the BDA soliton in a Fourier series, in which the amplitudes of the successive modes decrease, in keeping with a geometric progression law (see Ryzhov [25]). Wind-tunnel measurements led Borodulin and Kachanov [19] to an analogous claim that "The amplitudes of harmonic pulsations decay in almost rigorous conformity with a geometric progression law." In both cases, the common ratio of geometric progressions grows with a parameter determining the disturbance size and all the modes are in phase. Further theoretical studies reported by Bogdanova-Ryzhova and Ryzhov [30] and Ryzhov and Bogdanova-Ryzhova [31] provided evidence that the onset of random pulsations in the form of Hamiltonian chaos is inherent in the solitary-wave stage of boundary-layer motion.

Weaker disturbances, referred to as the TS eigenmodes, are scaled according to the triple-deck approach devised by Stewartson [32], Messiter [33], and Neiland [34] and independently extended to stability problems by Smith [35] and Zhuk and Ryzhov [36] in the limit of large Reynolds numbers. A set of governing equations in the framework of this asymptotic theory is of boundary-layer type, with the self-induced pressure not given in advance. Smith [37] showed that finite-time breakup is inherent, in general, in any boundary layer subject to viscous–inviscid interaction. Thus, well before the preceding soliton stage in the N and K paths to transition is reached, more new physics can come in play. A singularity in the velocity field derived by Smith [37] underlies (according to Li et al. [38], Smith et al. [39] and Bowles et al. [40]) a different scenario of a spiking process that includes the local development of pressure gradients in the normal-to-wall direction and vortex windup. The studies just cited lead to the extended Korteweg–de Vries (KdV) and BDA equations for the spike-soliton regions; however, these equations apply to the self-induced pressure, rather than the instantaneous displacement thickness. The computation of an initially nearly harmonic wave train by Bowles et al. [40] is broadly in agreement with the theoretical predictions; in particular, the normal-to-wall pressure gradients are consistent with those being caused by the streamline curvature. The finite-time breakup opens the way to a different approach to a spiky stage of the disturbance development discussed at length previously. This approach is not touched upon in the present study, which is focused on three-dimensional aspects of transition.

On the other hand, when processing computed skin-friction distributions, Ryzhov and Savenkov [41,42] observed extremely stable negative spikes within several central oscillation cycles of a strongly modulated triple-deck disturbance. On the contrary, positive wings of the same cycles are prone to the influence of short-length wiggles, giving rise to rapid distortions and breakdown of a vigorously growing wave packet. Reverse-flow bubbles are embedded in the stable negative skin-friction cycles when viewed in a frame of reference moving with the group velocity of the signal. It is remarkable that these local separation zones have no impact on the well-defined shape of the spikes. According to Ryzhov and Timofeev [43] and Ryzhov [44], the potential vortex/boundary-layer interaction as well as sound scattering produce similar wave packets, preserving their negative parts intact, despite explosive amplification in the downstream motion. Lower-amplitude wiggles were also seen by Bowles et al. [40] to corrupt a few pulsation cycles in the forepart of a nonlinear TS wave train. Thus, the onset of deep transition in the K route under discussion is most likely to be determined by various factors such as the type of flow, the operational mode (harmonic or pulse) of a perturbing source, three-dimensionalization of the wave pattern, the level of external turbulence in wind-tunnel tests, etc. As reported by Gaster [45], the wave packet enters an essentially nonlinear stage much faster than with a typical wave train, insofar as its linear development is accompanied by strong modulation of eigenmodes with different frequencies and wave numbers. Observations by Wygnanski et al. [46] revealed that a short time later, the wave packet turns into a turbulent spot, even though the surrounding boundary layer remains laminar.

The work by Moston et al. [47] cast doubts on the very nature of spikes in the K route to transition, which are inexorably associated with the BDA solitons from the preceding studies. The authors applied a multiscale technique to describe disturbances in the Blasius boundary layer in the far-downstream/high-frequency limit of the lower stability-branch (HFLB) regime, in which the growth rate is small. An intermediate diffusion sublayer sandwiched between the Stokes layer and the outer inviscid region played an important role in determining the two-dimensional disturbance pattern. In the authors' view, the occurrence of zones with closed streamlines in the frame of reference moving with the wave speed makes the validity of the HFLB solution not immediately clear, in spite of the fact that local separation bubbles appear in the computations by Bowles et al. [40] and by Ryzhov and Savenkov [42] showing a remarkable stability.

The focus in the current study is on truly nonlinear three-dimensional disturbances for which the diffusion sublayer is found to become purely passive. As noted earlier with reference to Kachanov [15,16], self-focusing leads to the strong dependence of spike parameters on all spatial coordinates. Accordingly, we start in Sec. II with presenting a pertinent set of unsteady three-dimensional asymptotic equations supplemented with limit conditions at infinity. The asymptotic description in Sec. III, as applied to one-dimensional solitary waves, illustrates the basic concepts of inviscid–inviscid interaction driving two-dimensional disturbances in the adjustment sublayer. Section IV contains the derivation of the wave dispersion law central to the whole study. In terms of the self-induced pressure, the wave dispersion is expressed through the Laplace operator, whatever the base flow. Solutions for free and forced linear oscillations are discussed in Sec. IV. Specific formulations set forth in Sec. V in terms of the instantaneous displacement thickness stem from the interaction law and result in new dynamical systems of the BDA and KdV types extended to two spatial variables. Oblique solitons and periodic nonlinear disturbances are indicated in Sec. VI for the Blasius boundary layer. A generalized Hirota function defined in Sec. VII yields a crossed-soliton solution for the near-wall jet. A sharp resonance features the two-line-soliton crossing.

II. Asymptotic Equations and Limit Conditions

Experimentally, the first nonlinear stage of the TS wave development in the K route to transition sets in fairly soon after the disturbance size has grown to a certain magnitude. The appropriate asymptotic as the Reynolds number

$$Re \to \infty$$

triple-deck scheme was independently invented by Stewartson [32], Neiland [34], and Messiter [33] in their studies on the trailing-edge flow and supersonic separation. Later, Smith [35] and Zhuk and Ryzhov [36] extended the theory to include hydrodynamic stability problems. A typical pressure rise in the triple-deck approach is of order ϵ^2 in terms of

$$\epsilon = Re^{-1/8}$$

In the lower viscous sublayer adjacent to a solid surface, the disturbance pattern obeys a set of Prandtl equations that involve the self-induced pressure to be determined simultaneously with the velocity field. When the self-induced pressure builds up beyond this limit and becomes as large as Δ^2, where this small parameter obeys inequalities

$$\epsilon \ll \Delta \ll 1 \tag{1}$$

a new asymptotic regime supersedes the triple deck. The viscous near-wall sublayer subdivides into two decks, with the upper one containing predominantly inviscid disturbances.†

Following Zhuk and Ryzhov [22] and Smith and Burggraf [23], we introduce the following scaled nondimensional variables in the adjustment sublayer

$$\bar{t} = \epsilon^4 \Delta^{-2} \tau_w^{-3/2} t \tag{2}$$

$$\bar{x} = 1 + \epsilon^4 \Delta^{-1} \tau_w^{-5/4} x \tag{3}$$

$$\bar{y} = \epsilon^4 \Delta \tau_w^{-3/4} y \tag{4}$$

$$\bar{z} = \epsilon^4 \Delta^{-1} \tau_w^{-5/4} z \tag{5}$$

and define the incompressible velocity field and self-induced pressure through

$$\bar{u} = \Delta(\tau_w^{1/4} u_1 + \Delta u_2 + \epsilon^4 \Delta^{-4} u_3 + \cdots) \tag{6}$$

$$\bar{v} = \Delta^3(\tau_w^{3/4} v_1 + \Delta v_2 + \epsilon^4 \Delta^{-4} v_3 + \cdots) \tag{7}$$

$$\bar{w} = \Delta(\tau_w^{1/4} w_1 + \Delta w_2 + \epsilon^4 \Delta^{-4} w_3 + \cdots) \tag{8}$$

$$\bar{p} = \Delta^2(\tau_w^{1/2} p_1 + \Delta p_2 + \epsilon^4 \Delta^{-4} p_3 + \cdots) \tag{9}$$

where τ_w is the wall skin friction in the initially undisturbed state. Here, the third-order terms are smaller than the second-order terms, provided that

$$\epsilon^{4/5} \ll \Delta$$

If, however,

$$\epsilon \ll \Delta \ll \epsilon^{4/5}$$

their positions in the asymptotic expansions (6–9) should be interchanged. The adjustment sublayer is thinner than the boundary layer, in which the normal-to-wall coordinate

$$\bar{y} = \epsilon^4 y_2$$

differs from Eq. (4) by a factor of Δ.

Substitution of Eqs. (2–5) and Eqs. (6–9) into the full system of Navier–Stokes equations yields

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 \tag{10}$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} + w_1 \frac{\partial u_1}{\partial z} = -\frac{\partial p_1}{\partial x} \tag{11}$$

$$\frac{\partial w_1}{\partial t} + u_1 \frac{\partial w_1}{\partial x} + v_1 \frac{\partial w_1}{\partial y} + w_1 \frac{\partial w_1}{\partial z} = -\frac{\partial p_1}{\partial z} \tag{12}$$

to leading order. We do not pursue the expansion scheme further, because higher-order terms giving small corrections to a solution of Eqs. (10–12) are determined from successive approximations, the complexity of which rapidly increases. For this reason, the subscript 1 will be subsequently omitted from labeling the desired functions. The system (10–12) is easily seen to result from the Prandtl equations for the classical unsteady three-dimensional boundary layer if the viscous stresses are supposed, according to scaling adopted in Eqs. (2–5), to be negligibly small. The self-induced pressure $p = p(t, x, z)$ has to be evaluated simultaneously with the velocity vector components. Most of the boundary layer and the inviscid zone outside are not considered in detail here; a routine analysis of these two sublayers can be found in the papers cited. As a result, the interaction law

$$p = L(A) \tag{13}$$

is derived from matching of solutions for all three decks in question. The interaction law relates the self-induced pressure to the instantaneous displacement thickness $-A(t, x, z)$, depending on the type of a boundary layer. The near-wall jet and the Blasius boundary layer are two typical examples.

A self-similar solution for the viscous motion that takes place along a solid flat plate in a quiescent fluid was found by Akatnov [48] and Glauert [49]. The excess pressure vanishes to zero at large distances from the near-wall jet. Owing to this fact, a second-order operator

$$L = -\frac{\partial^2 A}{\partial x^2} \tag{14}$$

that does not involve derivatives with respect to z specifies the interaction law indicated by Smith and Burggraf [23]. If the Blasius boundary layer is under consideration, an integral-differential operator

$$L = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^2 A/\partial \xi^2}{[(x-\xi)^2 + (z-\varsigma)^2]^{1/2}} \mathrm{d}\xi \mathrm{d}\varsigma \tag{15}$$

comes into play. An even more intricate interaction law

$$L = -\frac{1}{2\pi} \int_{-\infty} \int_{-\infty}^{\infty} \frac{\partial^2 A/\partial \xi^2}{[(x-\xi)^2 + (z-\varsigma)^2]^{1/2}} \mathrm{d}\xi \mathrm{d}\varsigma - \tilde{\Delta} \frac{\partial^2 A}{\partial z^2} \tag{16}$$

arises in the problem on large-sized crossflow disturbances in the boundary layer on a swept wing. According to Ryzhov and Terent'ev [50], a small parameter $\tilde{\Delta}$ depends on the momentum thickness

$$D_{zz} = \int_0^{\infty} W_0^2(y_2) \mathrm{d}y_2$$

based on crossflow W_0. Warping of stream surfaces in the lateral direction creates centrifugal forces to strike a balance with the excess pressure across the main deck.

Whatever the interaction law, a solution to the system of equations (10–12) should satisfy the limit conditions

$$u \to y + A(t, x, z) + \frac{1}{y} \int_{-\infty}^{x} \mathrm{d}\xi_1 \int_{-\infty}^{\xi_1} \frac{\partial^2 p(t, \xi_2, z)}{\partial z^2} \mathrm{d}\xi_2 \tag{17}$$

†The existence of the diffusion sublayer is ignored in the subsequent analysis because it plays a purely passive role in three-dimensional oscillation patterns under discussion.

$$v \to -\frac{\partial A}{\partial x}y - \frac{\partial A}{\partial t} - A\frac{\partial A}{\partial x} - \frac{\partial p}{\partial x} - \int_{-\infty}^{x} \frac{\partial^2 p(t,\xi_1,z)}{\partial z^2} d\xi_1 \tag{18}$$

$$w \to -\frac{1}{y}\int_{-\infty}^{x} \frac{\partial p(t,\xi_1,z)}{\partial z} d\xi_1 \tag{19}$$

at the upper reaches $y \to \infty$ of the adjustment sublayer, which follow from matching with an asymptotic representation of the corresponding functions in most of the boundary layer (see Kachanov et al. [29]). The boundary condition

$$v = 0 \quad \text{at } y = 0 \tag{20}$$

comes into operation for an effectively inviscid fluid at a flat plate, giving rise to a nonlinear problem in eigenvalues.

III. Two-Dimensional Disturbance Field

We begin with one-dimensional solitary waves driving two-dimensional disturbances with

$$w = \partial/z = 0 \tag{21}$$

In this case, the integral terms drop out of the right-hand side of Eqs. (17) and (19). Repercussions of the simplification made are of importance in several ways. When Eq. (21) holds, the limit conditions provide, as indicated by Zhuk and Ryzhov [22], an exact solution

$$u = y + A(t,x) \tag{22}$$

$$v = -\frac{\partial A}{\partial x}y - \frac{\partial A}{\partial t} - A\frac{\partial A}{\partial x} - \frac{\partial p}{\partial x} \tag{23}$$

to the set of inviscid boundary-layer equations. Thus, the adjustment sublayer is not required to be explicitly introduced in the asymptotic scheme considered by Smith and Burggraf [23]. Substitution of Eqs. (22) and (23) into the boundary condition (20) on a flat surface leads to

$$\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x} = -\frac{\partial p}{\partial x} \tag{24}$$

As evident from here, wave dispersion depends solely on the self-induced pressure specified in terms of the instantaneous displacement thickness by an interaction law.

For the near-wall jet, an operator L in Eq. (13) takes on the differential form (14), bringing into existence the KdV equation

$$\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x} = \frac{\partial^3 A}{\partial x^3} \tag{25}$$

which first appeared in the context of nonlinear shallow-water waves by Korteweg and de Vries [26]. A simple solution

$$A = -\frac{12k^2}{\cosh^2(\omega t + kx)} \tag{26}$$

$$\omega = 4k^3 \tag{27}$$

provides an explicit representation of the solitary wave of permanent form watched and described by Russel [51] as far back as 1838.

With assumption (21), the double integral on the right-hand side of Eq. (15) reduces to a Hilbert transform, and so the interaction law for two-dimensional disturbances in the Blasius boundary layer reads

$$L = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\partial A/\partial \xi}{x - \xi} d\xi \tag{28}$$

The resulting BDA integral-differential equation

$$\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\partial A/\partial \xi}{\xi - x} d\xi \tag{29}$$

was first derived by Benjamin [27] and Davis and Acrivos [28] in connection with the work on deepwater waves. Its algebraic solution found by Benjamin [27]

$$A = -\frac{4a}{a^2 + (x - ct)^2}, \qquad a = \frac{1}{c} \tag{30}$$

relates to the BDA solitons discussed from both the theoretical and experimental standpoints by Kachanov et al. [29] and in more detail by Kachanov [16]. Strong solitonlike signals are reported in the latter paper to also be observable in fully developed turbulent flows. A periodic solution

$$A = B_0 + \frac{B_1}{1 - q\cos(kx - \omega t)} \tag{31}$$

$$B_0 = \frac{1}{k}\left[\omega + \frac{k^2}{(1 - q^2)^{1/2}}\right] \tag{32}$$

$$B_1 = -2k(1 - q^2)^{1/2} \tag{33}$$

involving three arbitrary constants ω, k, and q also first appeared in [27]. It describes normal nonlinear TS waves. A passage to a limit

$$k \to 0, \qquad q \to 1$$

with the phase velocity $c = \omega/k$ being kept fixed in the periodic solution implies, in effect, the occurrence of a soliton (30) within each oscillation cycle. At an earlier triple-deck stage, the formation of extremely stable spikes in the central part of a modulated signal is seen in the computations by Ryzhov and Savenkov [41,42] and Ryzhov [44]. Similar spikes develop in oscillation cycles moving in the forepart of a nonlinear TS wave train computed by Bowles et al. [40].

IV. Wave Dispersion

The full system of nonlinear three-dimensional equations (10–12) subject to the limit conditions (17–19) as $y \to \infty$ is too complicated for its solution to be cast in an explicit form, even though all shear-stress terms are disregarded in a study of the adjustment sublayer. However, a law controlling wave dispersion is derivable by considering the small-amplitude limit of disturbances. An instructive lesson from the preceding analysis of two-dimensional wave patterns is that the nonlinear term $A\partial A/\partial x$ on the left-hand side of Eq. (24) exerts no impact on dispersion expressed through $\partial p/\partial x$. Of course, this does not mean the independence of the resulting displacement-thickness distribution from a balance stricken between nonlinearity and dispersion, but the two effects in question are of different natures. Therefore, dispersion should obey the same law no matter whether the disturbances are weak or strong. In other words, an expression for dispersion may be found by expanding Eqs. (10–12) in powers of the disturbance amplitude.

An initially equilibrium state is

$$u = y, \qquad v = w = p = A = 0 \tag{34}$$

throughout the adjustment sublayer. Let us introduce a positive parameter

$$0 < \delta < 1 \tag{35}$$

and write the following expansions

$$u = y + \delta u_1 + \delta^2 u_2 + \cdots \tag{36}$$

$$v = \delta v_1 + \delta^2 v_2 + \cdots \tag{37}$$

$$w = \delta w_1 + \delta^2 w_2 + \cdots \tag{38}$$

$$p = \delta p_1 + \delta^2 p_2 + \cdots \tag{39}$$

$$A = \delta A_1 + \delta^2 A_2 + \cdots \tag{40}$$

for the velocity field, excess pressure, and displacement thickness that represent deviations from the state of equilibrium brought about by a traveling wave system. Letting

$$\delta \to 0$$

and collecting the leading-order terms in Eqs. (10–12) gives the following set of linear homogeneous equations in the first approximation:

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 \tag{41}$$

$$\frac{\partial u_1}{\partial t} + y\frac{\partial u_1}{\partial x} + v_1 + \frac{\partial p_1}{\partial x} = 0 \tag{42}$$

$$\frac{\partial w_1}{\partial t} + y\frac{\partial w_1}{\partial x} + \frac{\partial p_1}{\partial z} = 0 \tag{43}$$

We proceed with establishing some properties inherent in the linear adjustment sublayer, with no regard to the limit conditions (17–19) at its upper reaches:

$$y \to \infty$$

Differentiation of Eq. (42) with respect to x leads to

$$\left(\frac{\partial}{\partial t} + y\frac{\partial}{\partial x}\right)\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial x} + \frac{\partial^2 p_1}{\partial x^2} = 0 \tag{44}$$

and differentiation of Eq. (43) with respect to z yields

$$\left(\frac{\partial}{\partial t} + y\frac{\partial}{\partial x}\right)\frac{\partial w_1}{\partial z} + \frac{\partial^2 p_1}{\partial z^2} = 0 \tag{45}$$

Combining the last two equations results in

$$\left(\frac{\partial}{\partial t} + y\frac{\partial}{\partial x}\right)F_1 + \frac{\partial v_1}{\partial x} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)p_1 = 0 \tag{46}$$

where a new desired function

$$F_1 = \frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} \tag{47}$$

We come to

$$\left(\frac{\partial}{\partial t} + y\frac{\partial}{\partial x}\right)\frac{\partial^2 u_1}{\partial x \partial y} + \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial x \partial y} = 0 \tag{48}$$

differentiating Eq. (44) with respect to y and differentiating Eq. (45) with respect to y leaves us with

$$\left(\frac{\partial}{\partial t} + y\frac{\partial}{\partial x}\right)\frac{\partial^2 w_1}{\partial y \partial z} + \frac{\partial^2 w_1}{\partial x \partial z} = 0 \tag{49}$$

The sum of Eqs. (48) and (49) is

$$\left(\frac{\partial}{\partial t} + y\frac{\partial}{\partial x}\right)\frac{\partial F_1}{\partial y} = 0 \tag{50}$$

on account of the continuity equation (41). It follows from Eq. (50) that

$$F_1 = F_1(t, x, z) \tag{51}$$

With allowance made for the boundary condition (20) imposed at a flat surface $y = 0$ with no perturbing source on it, Eq. (46) becomes

$$\frac{\partial F_1}{\partial t} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)p_1 \tag{52}$$

Neither of the two functions entering Eq. (52) depend on y; therefore, this equation remains valid regardless of a value of the normal-to-wall coordinate. Substitution into Eq. (46) of an expression for the Laplacian of p_1 obtained from Eq. (52) yields a simple equation that is integrated to specify the transverse velocity:

$$v_1 = -yF_1 \tag{53}$$

As Eq. (53) suggests, v_1 is a function of both velocity components u_1 and w_1 in a plane $y = \text{const}$ tangential to the solid surface.

The solution obtained holds within the adjustment sublayer, regardless of the outer conditions. Let us impose constraints following from Eqs. (17–19) on the general disturbance pattern. The first and the third of these limit conditions are linear and therefore consistent with reducing, in the first approximation, the full system of governing equations (10–12) to its linearized form in Eqs. (41–43). Differentiation of Eq. (17) with respect to x and Eq. (19) with respect to z leads to two expressions, the sum of which reads

$$\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} = \frac{\partial A_1}{\partial x} \quad \text{as } y \to \infty \tag{54}$$

whence

$$F_1 = \frac{\partial A_1}{\partial x} \tag{55}$$

on the strength of Eq. (51). The nonlinear term $A_1\partial A_1/\partial x$ entering Eq. (18) should be omitted to comply with an accuracy adopted in the leading-order analysis of the system of differential equations. Taking into account Eqs. (53) and (55), a linearized version of the limit condition (18) becomes

$$\frac{\partial^2 A_1}{\partial t \partial x} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)p_1 \tag{56}$$

The last relation supplemented with the interaction law controls the propagation of small-amplitude waves in the adjustment sublayer. The most important inference to be drawn from Eq. (56) is that dispersion of spatial disturbances obeys the Laplace operator of the self-induced pressure. This is nothing but the law of dispersion that we are looking for; it turns out to be homogeneous in both directions x and z. Mathematically, Eq. (56) implies a solvability condition for a set of linearized inviscid boundary-layer-type equations in three dimensions.

The Kadomtsev–Petviashvili (KP) equation (see Kadomtsev and Petviashvili [52])

$$\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial t} - \eta\frac{\partial \eta}{\partial x} - \frac{\partial^3 \eta}{\partial x^3}\right) = \frac{\partial^2 \eta}{\partial z^2} \tag{57}$$

is known in hydrodynamics to describe nonlinear solitary-wave disturbances varying in the lateral direction. In particular, an oblique KP soliton is given by

$$\eta = \frac{12k^2}{\cosh^2(\omega t + kx + mz)} \tag{58}$$

$$\omega = 4k^3 + \frac{m^2}{k} \tag{59}$$

A weak dependence on z (marked by the limit $\omega \to 4k^3$ as $k \to \infty$ and $m \to \infty$, with m^2/k fixed) makes the dispersion law not applicable to the case of interest in research on boundary-layer transition. The difference in the dispersion laws stems from the fact that the self-induced pressure creates a mechanism driving short-scaled large-sized solitary waves in a boundary layer of any kind, whereas in the context of long shallow-water waves, the pressure shows up as an external agency exerted on the free surface (see Akylas [53]). The homogeneity of dispersion arising from the action of the self-induced pressure appears to be natural from the physical standpoint. It should be emphasized that Eq. (56) equally relates to the near-wall jet, the conventional Blasius boundary layer, and the boundary layer on a swept wing, because the interaction law remained unspecified in the preceding reasoning. Asymmetry in x and z in the final formulation

$$\frac{\partial^2 A_1}{\partial t \partial x} = -L\left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial z^2}\right) \tag{60}$$

of the dispersion law in terms of the instantaneous displacement thickness results from an operator $L(A_1)$ in Eq. (13) that can take on one of the forms of Eqs. (14) and (15) or (16).

The arguments used to develop a linear solution can be inverted. To this end, let us start from Eq. (60) and first evaluate the instantaneous displacement thickness A_1 related to the problem in eigenvalues. The self-induced pressure p_1 comes from Eq. (13), thereby making both $\partial p_1/\partial x$ and $\partial p_1/\partial z$ in the inverse problem become known functions of t, x, and z. The evolution equation cast in Eq. (60) ensues from the linearized limit condition (18) by equating to zero all the y-independent terms on the right-hand side. Thus, the problem amounts to evaluating the velocity field from a set of linear equations (41–43). With $\partial p_1/\partial z$ prescribed in advance, the last equation separates from the others. Its direct integration determines the lateral velocity w_1 in an arbitrary plane $y = \text{const}$ parallel to the solid surface. The transverse velocity

$$v_1 = -y\frac{\partial A_1}{\partial x} \tag{61}$$

derives from Eq. (53) with F_1 substituted from Eq. (55). With v_1 and $\partial p_1/\partial x$ specified, Eq. (42) integrates to give the streamwise velocity u_1. What still remains to be shown is that the continuity equation (41) identically vanishes to zero with the velocity field found. This can be easily achieved by reiterating the arguments exploited previously when solving the direct problem. In fact, it is sufficient to notice that no constraint on the pressure distribution was imposed up to the point at which Eq. (56) came into operation. But it has been just the fundamental equation that provided, in the form of Eq. (60), the starting point for the proof of the inverse problem solvability.

V. Linear Oscillations

There is one more important issue to be resolved in the framework of the linear approach. This can be illuminated by introducing a particular solution

$$(u_1, v_1, w_1, p_1, A_1) = [\bar{U}(y), \bar{V}(y), \bar{W}(y), \bar{P}, \bar{A}]e^{\omega t + i(kx+mz)} \tag{62}$$

of the traveling-wave type into the preceding general analysis. From Eqs. (41–43), a set of the first-order differential equation

$$\frac{d\bar{V}}{dy} = -i(k\bar{U} + m\bar{W}) \tag{63}$$

supplemented with two finite relations

$$(\omega + iky)\bar{U} + \bar{V} = -ik\bar{P} \tag{64}$$

$$(\omega + iky)\bar{W} = -im\bar{P} \tag{65}$$

ensues to control the wave pattern. As it follows from here, the complex-valued amplitudes $\bar{U}$, $\bar{V}$, and $\bar{W}$ of the velocity vector components can be presented in terms of $\bar{P}$. Following the way of reasoning used at the end of the general analysis, we derive from Eq. (56) the first part

$$\bar{P} = \frac{i\omega k}{k^2 + m^2}\bar{A} \tag{66}$$

of the dispersion relation to link $\bar{P}$ and $\bar{A}$. As usual, the second part comes from a specific form of the interaction law (13). With the displacement thickness $-\bar{A}$ evaluated from Eq. (60), the self-induced pressure $\bar{P}$ can be regarded as a given function to be introduced into the right-hand sides of Eqs. (64) and (65). The normal-to-wall velocity

$$\bar{V} = -iky\bar{A} \tag{67}$$

and the lateral velocity

$$\bar{W} = -\frac{im}{\omega + iky}\bar{P} \tag{68}$$

are found in a straightforward manner from Eqs. (61) and (65), respectively. Then Eq. (64) determines the amplitude

$$\bar{U} = \bar{A} + \frac{im^2}{k(\omega + iky)}\bar{P} \tag{69}$$

of the streamwise velocity. It is easily seen that the disturbance field cast in Eqs. (67–69) satisfies the continuity equation (63). Notice that the disturbance field was obtained with no allowance made for the limit conditions at the upper reaches of the adjustment sublayer. For the traveling-wave type solution at hand, Eqs. (17) and (19) reduce to

$$\bar{U} \to \bar{A} + \frac{m^2}{k^2 y}\bar{P}, \qquad \bar{W} \to -\frac{m}{ky}\bar{P} \quad \text{as } y \to \infty$$

and are automatically met by Eqs. (68) and (69). Because the fluid in the adjustment sublayer is treated as effectively inviscid, both components

$$\bar{U} = -\frac{ik}{\omega}\bar{P}, \qquad \bar{W} = -\frac{im}{\omega}\bar{P} \tag{70}$$

do not vanish to zero on a flat plate $y = 0$.

To extend the preceding study to forced disturbances, let a small-amplitude vibrating device

$$y = \delta y_w(t, x, z) \tag{71}$$

be installed on an otherwise flat plate. Then the boundary condition

$$v_1 = \frac{\partial y_w}{\partial t} \quad \text{at } y = 0 \tag{72}$$

supersedes Eq. (20). In this case, the time derivative of F_1 obeys an equation

$$\frac{\partial F_1}{\partial t} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)p_1 - \frac{\partial^2 y_w}{\partial t \partial x} \tag{73}$$

with an additional term $-\partial^2 y_w/\partial t \partial x$ on the right-hand side, as compared with Eq. (52). As a consequence, Eq. (53) changes to

$$v_1 = -yF_1 + \frac{\partial y_w}{\partial t} \tag{74}$$

With these results in hand, we arrive at an equation

$$\frac{\partial^2 A_1}{\partial t \partial x} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)p_1 = -\frac{\partial^2 y_w}{\partial t \partial x} \tag{75}$$

controlling the linear disturbance radiation by an external agency. Following Ryzhov and Bogdanova-Ryzhova [31], we introduce a generalized displacement thickness

$$A_{1w} = A_1 + y_w \tag{76}$$

to cast Eq. (75) in a canonical form

$$\frac{\partial^2 A_{1w}}{\partial t \partial x} + L\left(\frac{\partial^2 A_{1w}}{\partial x^2} + \frac{\partial^2 A_{1w}}{\partial z^2}\right) = L\left(\frac{\partial^2 y_w}{\partial x^2} + \frac{\partial^2 y_w}{\partial z^2}\right) \tag{77}$$

containing only spatial derivatives of y_w on the right-hand side.

VI. Dynamical Systems

As noted in Sec. IV, nonlinearity and dispersion are different in nature. Therefore, a simplified approach was applied previously to establish the law of dispersion by solving a system of linear equations over the entire adjustment sublayer, taking into account the limit conditions at its upper edge. Now we are in a position to combine the two effects. A standard term $A\partial A/\partial x$ is known to feature truly nonlinear asymptotic theories. The incorporation of this term into the left-hand side of Eq. (56) yields

$$\frac{\partial}{\partial x}\left(\frac{\partial A}{\partial t}+A\frac{\partial A}{\partial x}\right)+\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial z^2}\right)=0 \tag{78}$$

if no vibrating device is placed on an otherwise flat plate. Notice that Eq. (78) is obtainable from equating to zero all the terms on the right-hand side of the limit condition (18) that do not include the normal-to-wall coordinate y. This general equation takes on different forms, depending on the type of the interaction law. Because the inviscid–inviscid interaction obeys one of the linear relations (14) and (15) or (16), the latter equation can be written as

$$\frac{\partial}{\partial x}\left(\frac{\partial A}{\partial t}+A\frac{\partial A}{\partial x}\right)=L\left(\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial z^2}\right) \tag{79}$$

Solitary and nonlinear periodic waves intrinsic to this dynamical system are of prime interest for research on deep transition. As usual, large-amplitude disturbances emitted by an external source are preceded by a much weaker oscillatory tongue moving in front of them (see Ryzhov and Bogdanova-Ryzhova [31]). The linear solution discussed previously provides an approximate description of pulsation cycles that comprise the tongue.

The preceding line of reasoning may be reinforced by considering the higher-order terms in the expansions (36–40). In the second-order approximation, we derive from Eqs. (10–12) a set of linear inhomogeneous equations:

$$\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial w_2}{\partial z}=0 \tag{80}$$

$$\frac{\partial u_2}{\partial t}+y\frac{\partial u_2}{\partial x}+v_2+\frac{\partial p_2}{\partial x}=-u_1\frac{\partial u_1}{\partial x}-v_1\frac{\partial u_1}{\partial y}-w_1\frac{\partial u_1}{\partial z} \tag{81}$$

$$\frac{\partial w_2}{\partial t}+y\frac{\partial w_2}{\partial x}+\frac{\partial p_2}{\partial z}=-u_1\frac{\partial w_1}{\partial x}-v_1\frac{\partial w_1}{\partial y}-w_1\frac{\partial w_1}{\partial z} \tag{82}$$

The left-hand sides of Eqs. (80–82) are similar to those in Eqs. (41–43); therefore, the first-order analysis can be extended to treat the second-order functions.

Differentiation of Eq. (81) with respect to x gives

$$\left(\frac{\partial}{\partial t}+y\frac{\partial}{\partial x}\right)\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial x}+\frac{\partial^2 p_2}{\partial x^2} =-\frac{\partial}{\partial x}\left(u_1\frac{\partial u_1}{\partial x}+v_1\frac{\partial u_1}{\partial y}+w_1\frac{\partial u_1}{\partial z}\right) \tag{83}$$

whereas differentiation of Eq. (82) with respect to z leaves us with

$$\left(\frac{\partial}{\partial t}+y\frac{\partial}{\partial x}\right)\frac{\partial w_2}{\partial x}+\frac{\partial^2 p_2}{\partial z^2}=-\frac{\partial}{\partial z}\left(u_1\frac{\partial w_1}{\partial x}+v_1\frac{\partial w_1}{\partial y}+w_1\frac{\partial w_1}{\partial z}\right) \tag{84}$$

Combining Eqs. (83) and (84) yields an expression

$$\left(\frac{\partial}{\partial t}+y\frac{\partial}{\partial x}\right)F_2+\frac{\partial v_2}{\partial x}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)p_2=Q_1(t,x,y,z) \tag{85}$$

Here, the second-order function F_2 is introduced by

$$F_2=\frac{\partial u_2}{\partial x}+\frac{\partial w_2}{\partial z}=-\frac{\partial v_2}{\partial y} \tag{86}$$

on the strength of Eq. (80) and the right-hand side, Q_1 reads

$$Q_1=-\frac{\partial}{\partial x}\left(u_1\frac{\partial u_1}{\partial x}+v_1\frac{\partial u_1}{\partial y}+w_1\frac{\partial u_1}{\partial z}\right) -\frac{\partial}{\partial z}\left(u_1\frac{\partial w_1}{\partial x}+v_1\frac{\partial w_1}{\partial y}+w_1\frac{\partial w_1}{\partial z}\right) \tag{87}$$

In view of the boundary condition (20), a relation

$$\frac{\partial F_2}{\partial t}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)p_2=Q_1(t,x,0,z) \tag{88}$$

should be imposed on F_2 and p_2 at a flat plate $y=0$.

Consider the limit conditions at infinity, as $y\to\infty$, for the second-order functions involved in Eqs. (36–40). By virtue of linearity, Eq. (17) results in

$$u_2\to A_2+\frac{1}{y}\int_{-\infty}^{x}\mathrm{d}\xi_1\int_{-\infty}^{\xi_1}\frac{\partial^2 p_2(t,\xi_2,z)}{\partial z^2}\mathrm{d}\xi_2 \tag{89}$$

For the same reason, Eq. (19) becomes

$$w_2\to-\frac{1}{y}\int_{-\infty}^{x}\frac{\partial p_2(t,\xi_1,z)}{\partial z}\mathrm{d}\xi_1 \tag{90}$$

whereas a term $-A_1\partial A_1/\partial x$ independent of y enters the right-hand side of the condition

$$v_2\to-y\frac{\partial A_2}{\partial x}-\frac{\partial A_2}{\partial t}-\frac{\partial p_2}{\partial x}-\int_{-\infty}^{x}\frac{\partial p_2(t,\xi_1,z)}{\partial z}\mathrm{d}\xi_1-A_1\frac{\partial A_1}{\partial x} \tag{91}$$

as a consequence of the nonlinear character of Eq. (18). From Eqs. (89) and (90), the limit condition for F_2 reads

$$F_2\to\frac{\partial A_2}{\partial x}\quad \text{as } y\to\infty \tag{92}$$

However, the crucial point is that both momentum equations (81) and (82), like the corresponding original nonlinear equations (11) and (12), are satisfied only to within the terms proportional to y^1 and y^0 in magnitude of the normal-to-wall distance $y\to\infty$. The higher-order terms containing negative powers of y do not cancel out. Hence, the same accuracy should be kept in the analysis of Eq. (85), where

$$Q_1\to-\frac{\partial}{\partial x}\left(A_1\frac{\partial A_1}{\partial x}\right)\quad \text{as } y\to\infty \tag{93}$$

With allowance made for the last two limits, we have

$$\frac{\partial v_2}{\partial x}\to-\left(\frac{\partial}{\partial t}+y\frac{\partial}{\partial x}\right)\frac{\partial A_2}{\partial x}-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)p_2-\frac{\partial}{\partial x}\left(A_1\frac{\partial A_1}{\partial x}\right) \tag{94}$$

This is nothing but the x derivative of the limit condition (91). Thus, Eq. (85) meets both the boundary condition at $y=0$ and the limit conditions as $y\to\infty$, provided that the constraint (88) is imposed on F_2 and p_2. Obviously, the evolution equation

$$\frac{\partial^2 A_2}{\partial t\partial x}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)p_2+\frac{\partial}{\partial x}\left(A_1\frac{\partial A_1}{\partial x}\right)=0 \tag{95}$$

offers an extension of this constraint that complies with the leading-order approximation.

Turning to the inverse problem, we start from the second-order evolution Eq. (95) cast in the form

$$\frac{\partial^2 A_2}{\partial t\partial x}+L\left(\frac{\partial^2 A_2}{\partial x^2}+\frac{\partial^2 A_2}{\partial z^2}\right)+\frac{\partial}{\partial x}\left(A_1\frac{\partial A_1}{\partial x}\right)=0 \tag{96}$$

to first evaluate the instantaneous displacement thickness $-A_2$. The self-induced pressure comes from the interaction law

$$p_2 = L(A_2) \tag{97}$$

thereby making both $\partial p_2/\partial x$ and $\partial p_2/\partial z$ become prescribed functions of t, x, and z in the momentum equations (81) and (82). The full system (80–82) determines the second-order velocity field u_2, v_2, and w_2 expressed in terms of F_2 and v_2 by means of Eqs. (85) and (86). Substitution of the Laplace operator of p_2 obtained from Eq. (95) into Eq. (85) yields an expression

$$\left(\frac{\partial}{\partial t} + y\frac{\partial}{\partial x}\right)F_2 + \frac{\partial v_2}{\partial x} - Q_1(t,x,y,z) = \frac{\partial^2 A_2}{\partial t \partial x} + \frac{\partial}{\partial x}\left(A_1\frac{\partial A_1}{\partial x}\right) \tag{98}$$

that holds true for any y. On account of Eqs. (92) and (93), it converges to the limit condition

$$\frac{\partial v_2}{\partial x} + y\frac{\partial^2 A_2}{\partial x^2} \to 0 \tag{99}$$

following from Eq. (91), with the instantaneous displacement thickness $-A_2$ and the self-induced pressure p_2 related through Eq. (95). Thus, Eq. (85) automatically meets the limit condition at the upper reaches of the adjustment sublayer. On the other hand, Eq. (98) reduces to Eq. (88) on the strength of the boundary condition (20) prescribed at a flat surface $y = 0$.

It should be emphasized that the explicit dependence of Q_1 on the time and spatial coordinates played no role in the preceding consideration. This property allows us to extend the results obtained to higher-order terms in the expansion (36–40) underlying the nonlinear velocity and pressure fields in the adjustment sublayer.

VII. Blasius Boundary layer

The boundary layer on a flat plate is the cornerstone of hydrodynamic stability theory, be it a linear stage or a truly nonlinear regime typical of deep transition. The integral-differential operator (15) applies to specify the type of inviscid–inviscid interaction between the outer flow and the adjustment sublayer. The general dynamical system defined in Eq. (79) turns to

$$\frac{\partial}{\partial x}\left(\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x}\right) = \frac{1}{2\pi}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^2 A/\partial \xi^2}{[(x-\xi)^2 + (z-\varsigma)^2]^{1/2}}\mathrm{d}\xi \mathrm{d}\varsigma\right] \tag{100}$$

This evolution equation is too complicated for existing analytical methods to be employed in a search for the general type of solitary and nonlinear periodic waves propagating in a boundary layer. However, oblique nonlinear disturbances can be obtained in a straightforward manner by including the dependence on the second coordinate in the one-dimensional solutions (30–33) by Benjamin [27]. It is easily verified that a function

$$A = B_0 + \frac{B_1}{1 - q\cos(kx + mz - \omega t)} \tag{101}$$

$$B_0 = \frac{1}{k}\left[\omega + \frac{k(k^2+m^2)^{1/2}}{(1-q^2)^{1/2}}\right] \tag{102}$$

$$B_1 = -2(k^2+m^2)^{1/2}(1-q^2)^{1/2} \tag{103}$$

periodic in both the streamwise and lateral directions satisfies Eq. (100). Four arbitrary constants ω, k, m, and q can be used to make the theory fit observational data. Experience gained in the work on one-dimensional disturbances is encouraging (see Kachanov et al. [29] and Kachanov [16]).

In the case of weak oscillations, the instantaneous displacement thickness tends to

$$A \to \frac{\omega}{k} + (k^2+m^2)^{1/2} - 2q(k^2+m^2)^{1/2}\cos(kx + mz - \omega t) + \cdots \tag{104}$$

This is nothing but an oblique linear TS wave with the amplitude factor $q \to 0$. Thus, Eqs. (101–103) may be regarded as an oblique nonlinear TS wave propagating, in general, against a nonzero background fixed by a value of B_0. If the boundary layer is initially unperturbed, the background vanishes to zero. Hence, we have a dispersion relation:

$$c = -(k^2+m^2)^{1/2}(1-q^2)^{-1/2} \tag{105}$$

where the phase velocity $c = \omega/k$ depends on both wave numbers k and m as well as the amplitude factor q. In consequence, Eq. (101) becomes

$$A = \frac{B_1}{1 - q\cos(kx + mz - \omega t)} \tag{106}$$

With $q \to 0$, Eq. (105) converges to the short-wavelength limit

$$c \to -(k^2+m^2)^{1/2} \tag{107}$$

inherent both in the triple-deck theory (see Smith [35] and Zhuk and Ryzhov [36]) and in the present approach, culminating in the expression (66) supplemented with interaction law (15).

The occurrence of an oblique soliton within each oscillation cycle of the large-sized periodic wave [Eqs. (101–103)] can be brought to light by using a line of reasoning similar to that described at the end of Sec. III in connection with one-dimensional disturbances. To this end, let us pass to a short-scaled limit as $k \to 0$ and

$$c \to c_0 = \text{const} \tag{108}$$

$$\frac{m}{k} \to m_0 = \text{const} \tag{109}$$

Under these conditions, the streamwise wave number

$$k \to -c_0(1+m_0^2)^{-1/2}(1-q^2)^{1/2} \tag{110}$$

on the strength of Eq. (105), which leads to a requirement $q \to 1$ or

$$\frac{1-q^2}{k^2} \to q_0^2(1+m_0^2) \tag{111}$$

With $k \to 0$ and $q \to 1$, we can write a Taylor series expansion

$$1 - q\cos(kx + mz - \omega t) = (1-q)\left[1 + \frac{1}{q_0^2(1+m_0^2)}(x + m_0 z - c_0 t)^2\right] + \cdots \tag{112}$$

to be substituted into Eq. (106). The final result reads

$$A = -\frac{4a_0(1+m_0^2)^{1/2}}{a_0^2 + (x + m_0 z - c_0 t)^2} \tag{113}$$

where three constants a, c_0, and m_0 are related through

$$ac_0 = -(1+m_0^2)^{1/2} \tag{114}$$

This is an oblique solitary wave intrinsic to the dynamical system cast in Eq. (100). With $m_0 = 0$, Eqs. (113) and (114) reduce to the one-dimensional soliton (30) by Benjamin [27] that propagates in the mainstream direction. If, in general, $m_0 \neq 0$, both the phase velocity and the amplitude of the oblique soliton depend on the lateral wave number. On the other hand, the functional dependence of A in Eqs. (113) and (114) on time and spatial coordinates closely parallels that in Eq. (30). Accordingly, the characteristic dimensions of all large-sized short-scaled coherent structures developing in the intermediate adjustment sublayer are of the same order in magnitude. This structural analogy is most likely to offer a simple explanation for a remarkable success achieved by Kachanov et al. [29] and Kachanov

[16] in a comparison between theoretical predictions and observational data. Theoretically, the one-dimensional BDA soliton (30) is at the bottom of this work, whereas experimental data are based on the wind-tunnel measurements, which inevitably involve disturbance-field variations in the spanwise direction. Thus, the experimental finding by Kachanov et al. [29] and Kachanov [16] relate to the BDA-like oblique solitons (113) and (114), all of which are of similar shape, irrespective of the obliqueness angle fixed by m_0. It is because of the structural similarity that the observed solitary-wave forms appear to be so well-suited for presenting the evolution of one-dimensional BDA solitons.

Probably the most decisive conclusion follows from the spectral decomposition of experimentally recorded disturbances transitioning to turbulence. As noted in the Introduction, observational data led Borodulin and Kachanov [19] to claim that the amplitudes of the successive harmonic modes decay "in almost rigorous conformity with a geometric progression law." This statement is substantiated by a Fourier series expansion

$$\frac{2(1-q^2)^{1/2}}{1-q\cos(kx+mz-\omega t)} = 4\pi\left[\frac{1}{2}+\sum_{n=1}^{\infty} r^n \cos\Big[n(kx+mz-\omega t)\Big]\right]$$

of the function that specifies a large-sized periodic-wave train [Eqs. (101–103)] with a solitonlike peak embedded in each oscillation cycle. The amplitudes of the successive modes here do actually obey the geometric progression law with the common ratio

$$r = \frac{1-(1-q^2)^{1/2}}{q}$$

fixed in wind-tunnel tests, no matter the value of the lateral wave number m (or m_0). For more detailed comparisons of experimental findings and theoretical evidence, see Kachanov [16].

The adjustment sublayer was treated previously as a site at which high-amplitude periodic and solitary waves were given birth. However, apart from accommodating large-sized coherent structures, the adjustment sublayer is endowed, according to Ryzhov and Bogdanova-Ryzhova [31], with another important role related to irregular pulsations with erratic sequences of amplitudes, phases, and characteristic times. Both properties are interconnected and inseparable. The erratic disturbances derive from the wave/vortex eigenmode interaction maintained by centrifugal forces due to stream-surface curvature. Even though the irregular pulsations are an integral part of deep transition , they will be touched upon only briefly. The crisscross interaction brought about on a swept wing by warping of stream surfaces in the lateral direction obeys Eq. (16). Centrifugal forces can also be created in the disturbance field, independently of the spanwise coordinate. In this case, a term with $\partial^2 A/\partial x^2$ comes in place of $\partial^2 A/\partial z^2$ on the right-hand side of Eq. (16), rendering the interaction law much simpler. A study by Ryzhov and Bogdanova-Ryzhova [31] gives an idea of low-dimensional turbulence emerging at the end of deep transition before the fully developed turbulent flow sets in.

What is more, a set of inviscid Eqs. (10–12) also applies to turbulent regimes in an investigation into well-organized large-amplitude pulses, for the Reynolds shear stresses can be ignored within the adjustment sublayer. Experimental findings summarized by Kachanov [16] provide strong support for this view. His work presents oscilloscope traces of the streamwise velocity components in a transitional boundary layer, experiencing the K route to breakdown, and the corresponding traces of the ensemble averaged structures in the fully developed turbulent boundary layer obtained by means of a conditional sampling technique. A comparison of these records shows a close resemblance of the shapes of spikes in both regimes, the spike in the transitional boundary layer being almost indistinguishable from the BDA soliton. The resemblance demonstrated by Kachanov [16] becomes even more striking if one takes into account that the conditional sampling technique, as a rule, gives underestimated amplitudes owing to the influence of noise produced by background turbulence inevitable in wind-tunnel measurements. The preceding remarks on oblique disturbances are relevant to turbulent flow data.

VIII. Near-Wall Jet

The inviscid–inviscid interaction that takes place in a jet adjacent to a solid surface is governed by a simple differential operator defined in Eq. (14). In this case, Eq. (79) integrates with respect to x and results in a third-order equation:

$$\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial z^2}\right) = \Delta\left(\frac{\partial A}{\partial x}\right) \tag{115}$$

where Δ, as usual, designates the Laplace operator. The difference between Eqs. (57) and (115) primarily stems from the fact that the KP equation controls the long-wave surface motion, which is weakly nonlinear, weakly dispersive, and weakly two-dimensional, with all three effects being of the same order in magnitude. On the contrary, both spatial coordinates entering Eq. (115) are equally scaled in Eqs. (3) and (5), insofar as the self-induced pressure creates a mechanism driving short-wavelength disturbances in the adjustment sublayer of the jet.

As distinct from nonlinear disturbances in the Blasius boundary layer governed by Eq. (100), solitary waves in the near-wall jet can be obtained in a regular way by introducing a generalized Hirota function. Originally, the Hirota function has been devised in connection with the KdV equation, but it later became clear that this function was also directly applicable to the KP equation (see Hirota [54], Drazin and Johnson [55], and Ablowitz and Clarkson [56]). However, an extension is required when we turn to Eq. (115), in which both spatial coordinates play an equal role. The generalized Hirota function f in an expression

$$A = -12\nabla^2 \log f \tag{116}$$

allowing for the Laplace operator of $\partial A/\partial x$ on the right-hand side of Eq. (115) differs from its classic version

$$A = -12\frac{\partial^2}{\partial x^2}\log f \tag{117}$$

by the derivative $\partial^2 f/\partial z^2$ entering Eq. (116). Substitution of Eq. (116) into Eq. (115) results in an equation

$$\frac{2}{f^3}\frac{\partial f}{\partial t}(\nabla f)^2 - \frac{1}{f^2}\left[2\left(\nabla f\cdot\nabla\left(\frac{\partial f}{\partial t}\right)\right)+\frac{\partial f}{\partial t}\Delta f\right]+\frac{1}{f}\Delta\left(\frac{\partial f}{\partial t}\right) + \frac{\partial}{\partial x}\left[-\frac{3}{f^2}(\Delta f)^2+\frac{4}{f^2}\Big(\nabla f\cdot\nabla(\Delta f)\Big)-\frac{1}{f}\Delta(\Delta f)\right]=0 \tag{118}$$

that seems formidable compared with the initial third-order equation for the displacement thickness. However, there exists a simple technique of deriving multisoliton solutions of Eq. (115) that is based on the assumption that f can be expanded in integral powers of a small parameter δ. Accordingly, we write an expansion

$$f = 1 + \delta f_1(t,x,z) + \delta^2 f_2(t,x,z) + \cdots \tag{119}$$

and substitute it into Eq. (118). The linear in δ terms yield

$$\frac{\partial f_1}{\partial t} - \Delta\left(\frac{\partial f_1}{\partial x}\right) = 0 \tag{120}$$

As has been pointed out by Hirota [54], the series terminates after a finite number of terms; therefore, δ may be chosen arbitrarily (of necessity, not small).

To find the one-soliton solution, let $\delta = 1$ and

$$f = 1 + f_1(t,x,z) = 1 + e^{2\eta} \tag{121}$$

$$\eta = \omega t + kx + mz \tag{122}$$

A dispersion relation

$$\omega = 4k(k^2 + m^2) \tag{123}$$

to connect ω with both wave numbers k and m ensues from Eq. (120). But the substitution of Eqs. (121) and (122) into Eq. (118) gives rise to two additional sets of terms proportional to $e^{4\eta}$ and $e^{6\eta}$. It is easily verified that the terms of the first set cancel out on the strength of Eq. (123), whereas the terms of the second set identically vanish to zero. Thus, Eqs. (121–123) provide an exact solution to Eq. (118) that defines an oblique soliton

$$A = -\frac{12(k^2 + m^2)}{\cosh^2(\omega t + kx + mz)} \tag{124}$$

characteristic of Eq. (115). Its amplitude is specified by the sum $k^2 + m^2$ quadratic in both wave numbers, rather than by the streamwise wave number k squared. In the limiting case $m = 0$, the frequency ω in Eqs. (59) and (123) equals the same value $\omega = 4k^3$ intrinsic to the KdV soliton (26) and (27). Thus, the two solitary-wave solutions under comparison converge to the KdV limit, no matter what the driving mechanism.

In search for two-soliton solutions, we truncate Eq. (119) after the second-order term, postulating

$$f = 1 + f_1(t, x, z) + f_2(t, x, z) \tag{125}$$

where

$$f_1 = e^{2\eta_1} + e^{2\eta_2} \tag{126}$$

$$\eta_1 = \omega_1 t + k_1 x + m_1 z + \delta_1 \tag{127}$$

$$\eta_2 = \omega_2 t + k_2 x + m_2 z + \delta_2 \tag{128}$$

is an extension of Eqs. (121) and (122). Two dispersion relations

$$\omega_1 = 4k_1(k_1^2 + m_1^2) \tag{129}$$

$$\omega_2 = 4k_2(k_2^2 + m_2^2) \tag{130}$$

similar to Eq. (123), follow from Eq. (120). The next term

$$f_2 = Te^{2(\eta_1+\eta_2)} \tag{131}$$

does not depend on $e^{4\eta_1}$ and $e^{4\eta_2}$. Rather, tedious algebra shows that

$$T = \frac{P_1(k_1 - k_2)^2 + P_2(k_1 - k_2)(m_1 - m_2) + P_3(m_1 - m_2)^2 + Q_1(k_1 - k_2) + Q_2(m_1 - m_2) + R}{D} \tag{132}$$

Here, the coefficients

$$P_1 = 3k_1k_2(k_1 + k_2) \tag{133}$$

$$P_2 = 3k_1k_2(m_1 + m_2) \tag{134}$$

$$P_3 = 2m_1m_2(k_1 + k_2) \tag{135}$$

remain finite as

$$k_1 \to k_2, \qquad m_1 \to m_2$$

The quantities

$$Q_1 = -(k_1 + k_2)(k_1m_2^2 - k_2m_1^2) \tag{136}$$

$$Q_2 = 2(k_1^3m_2 - k_2^3m_1) - 3m_1m_2(k_1m_2 - k_2m_1) - (k_1m_2^3 - k_2m_1^3) \tag{137}$$

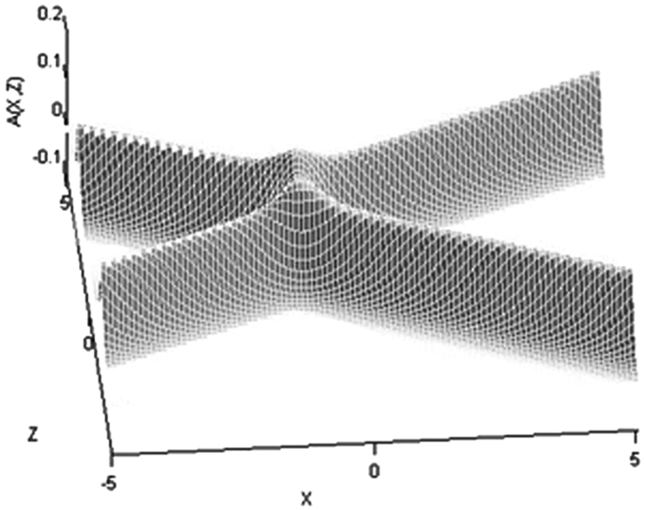

Fig. 1 Interaction of two line solitons results in a visible phase shift of both branches of each wave.

tend to zero linearly with both differences $k_1 - k_2$ and $m_1 - m_2$, whereas

$$R = -4(k_1m_2 - k_2m_1)(k_1^2m_2 - k_2^2m_1) \tag{138}$$

is quadratic in $k_1 - k_2$ and $m_1 - m_2$. Because the denominator

$$D = [3k_1k_2(k_1 + k_2) + 2m_1m_2(k_1 + k_2) + k_1m_2^2 + k_2m_1^2][(k_1 + k_2)^2 + (m_1 + m_2)^2] \tag{139}$$

approaches a finite value, T determines a second-order correction to Eqs. (126–128).

In the particular case of two solitons set at right angles to the oncoming stream, both lateral wave numbers $m_1 = m_2 = 0$. Then Eq. (125) reduces to a solution

$$f = 1 + e^{2\eta_1} + e^{2\eta_2} + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{2(\eta_1+\eta_2)} \tag{140}$$

of the KdV equation found by Hirota [54]. A pair of crossed solitons propagating in the mainstream direction is specified by the conditions

$$k_2 = k_1 = k, \qquad m_2 = -m_1 = -m$$

which result in

$$f = 1 + e^{2(\omega t+kx+mz)} + e^{2(\omega t+kx-mz)} - \frac{3m^2(k^2 + m^2)}{k^2(3k^2 - m^2)} e^{4(\omega t+kx)} \tag{141}$$

The exponent of the last term on the right-hand side of Eq. (141) that is brought about by the interaction of two line solitons does not depend on the lateral coordinate, whereas

$$\omega = 4k(k^2 + m^2) \tag{142}$$

varies with m on the strength of Eqs. (129) and (130). A sharp resonance occurs at $m = \pm\sqrt{3}k$. An analogous resonance is also inherent in a two-line-soliton solution of the KP equation (see, for example Ablowitz and Clarkson [56]). Figures 1 and 2 illustrate two different aspects of a typical soliton crossing. The instantaneous

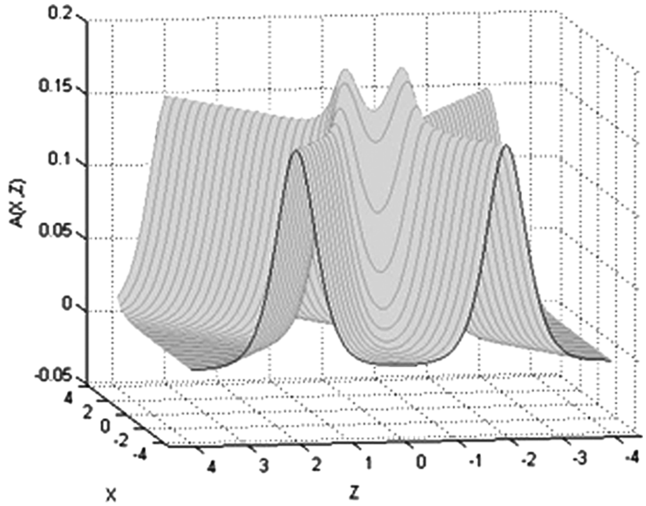

Fig. 2 Front view of the complex spatial shape of the interaction region between two line solitons inclined at angles $\tan\alpha_1 = 2$ and $\tan\alpha_2 = -2$ to the mainstream direction.

displacement thickness A is shown in a moving frame of reference $X = x + (\omega/k)t$, with $Z = z$ aligned with the x axis. The values $\tan\alpha_1 = 2$ and $\tan\alpha_2 = -2$ are used in the computation. A phase shift between both branches of either of two solitons is seen in Fig. 1. Their interaction results in a complex-shaped central region (Fig. 2).

IX. Conclusions

The key inference to be drawn from the present study is that dispersion of large-sized short-scaled disturbances in transitional boundary-layer-type flows is brought about by the self-induced pressure p and, for this reason, depends on both spatial coordinates located in a plane tangent to a solid surface. The Laplace operator of p yields an explicit expression for the wave dispersion. Asymmetry in the final formulation of the dispersion law as a function of the instantaneous displacement thickness $-A$ stems from a relation connecting p and A in the process of inviscid–inviscid interaction. An adjustment sublayer sandwiched between the main body of the boundary layer and the thin viscous region adjacent to the wall accommodates large-sized disturbances at the heart of this interaction. A two-dimensional extension of the BDA equation provides a pertinent dynamical system as applied to the Blasius boundary layer transitioning to turbulence. An analogous extension of the famous KdV equation leads to the dynamical system intrinsic to the near-wall jet. In the latter case, a technique employing a generalized Hirota function allows a solution for two crossed-line solitons to be cast in an explicit form. It differs significantly from a similar solution to the KP equation known from the shallow-water wave theory as an alternative two-dimensional extension of the KdV dynamical system. Though the two-dimensional evolution equations are distinctive, they converge to the same one-dimensional KdV limit. According to the experimental data reported by Kachanov [16], well-organized solitonlike coherent structures are also typical of fully developed turbulent boundary layers.

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